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SOME NON-QUASIREFLEXIVE SPACES HAVING UNIQUE ISOMORPHIC PREDUALS[†]

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ABSTRACT

It is shown that the dual spaces of certain James-Lindenstrauss spaces are spaces which are non-quasireflexive but have unique isomorphic preduals.

A Banach space Y is said to be a predual of a Banach space X if Y^*, the dual space of Y, is isomorphic (linearly homeomorphic) to X. We note that X may have a predual even though there may *not* exist any Banach space Z such that $Z^* = X$ isometrically (see [4]). A Banach space X is said to have a unique predual if X has a predual and all preduals are mutually isomorphic.

The following possible three cases actually occur. 1) X does not have any predual: for example, if X is c_0, $C[0,1]$ or $L_1[0,1]$ (see e.g. [2]). 2) X has a unique predual: This happens when X is quasireflexive, that is, the canonical embedding of X into the second dual space X^{**} has finite co-dimension (see [3]). 3) X has many non-isomorphic preduals: for example, if X is l_1, l_∞ or $L_\infty[0,1]$ (see e.g. [1] and [10]).

It is interesting to find conditions on X which imply uniqueness (or non-uniqueness) of preduals of X. For instance, as mentioned in 2), quasireflexivity of X implies that X has a unique predual; however, the converse of this seems to be an open question. In this paper, we will show that the converse is not true and that the dual spaces of certain James-Lindenstrauss spaces supply such examples.

Before we state our result, let us discuss notation. Capital letters X, Y, Z, A, B, etc., will always denote Banach spaces and small letters x, y, z, a, b, etc., will denote elements of Banach spaces. We write $X \sim Y$ if X is isomorphic to Y, and write $X \oplus Y$ for the direct sum of X and Y. We always regard X and X^* as subspaces, respeetively, of X^{**} and X^{***} in the canonical way. If A is a

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subspace of X, $A^\perp$ denotes the annihilator of A in X^*, if A is a subspace of X^*, then $A_\perp$ denotes the set of elements in X annihilated by A.

Our result can be stated as follows:

THEOREM. *Suppose a Banach space X with scalar field F (real or complex) satisfies the following three conditions,*

a) *X is separable,*

b) *$X^{**} = X \oplus A$ and $A \sim l_1$,*

c) *$X \oplus F \sim X$.*

Then X^* has a unique predual.

PROOF. We wish to point out that there exist Banach spaces with the above three properties. Let Y_0 be the James-Lindenstrauss space given in the Theorem in [9] (see also Theorem 1 in [8]) with the property that Y_0 has a basis and $Y_0^{**} = Y_0 \oplus A$ where $A \sim l_1$. The Banach space $X = Y_0 \oplus l_2$ satisfies the above three properties. (We understand that if one follows Lindenstrauss' construction of Y_0, one sees that Y_0 contains l_2 as a complemented subspace. Therefore Y_0 itself satisfies the above three conditions.)

If Y is a predual of X^*, then Y is isomorphic to a subspace X^{**} which is total over X^* and minimal with respect to the property of being total over X^* ([5]). Therefore we identify Y as a subspace of X^{**}. Y being minimal total implies that $X^{***} = X^* \oplus Y^\perp$. Similarly we have $X^{***} = X^* \oplus X^\perp$. Let P be the projection of X^{***} onto X^* with respect to the decomposition $X^{***} = X^* \oplus Y^\perp$. Let $S = P|_{X^\perp}; S: X^\perp \to X^*$. It is not difficult to see that

$$Y^\perp = \{z - Sz : z \in X^\perp\}. \tag{1}$$

Note that $X^\perp = (X^{**}/X)^* \sim A^* \sim l_1^* = l_\infty$, and observe that the weak* topology on X^{***} restricted to X^* is the same as the weak topology on X^* and that the weak* topology on X^{***} restricted to $X^\perp$ is the same as the weak* topology on $X^\perp$, where $X^\perp$ is regarded as the dual space of A.

We wish to show that if X^* is given the weak topology, and $X^\perp$ is given the weak* topology as the dual space of A, then S is continuous (we write S is w^*-w continuous). S is w^*-w continuous if and only if $S|_{B_{X^\perp}}$ is w^*-w continuous where $B_{X^\perp}$ is the unit ball of $X^\perp$. Since $X^\perp \sim A^*$ and $A \sim l_1$ which is separable, $B_{X^\perp}$ with the weak* topology is metrizable. Therefore S is w^*-w continuous if and only if $S|_{B_{X^\perp}}$ is sequentially w^*-w continuous. Let $z_n \in B_{X^\perp}$ and $z_n \xrightarrow{w^*} z \in B_{X^\perp}$. To complete the proof of continuity it is sufficient to show that every subsequence of $\{z_n\}$ has a subsequence $\{z_{n_\nu}\}$ such that $Sz_{n_\nu} \xrightarrow{w} Sz$ in X^*. By Grothendieck's theorem [7], since $X^\perp \sim l_\infty$ and X^* is separable (X^{**} is separable), S is a weakly compact operator, i.e., the weak closure of $S(B_{X^\perp})$ is weakly compact in X^*. Therefore, there exist a subsequence $\{z_{n_\nu}\}$ and a point u in X^* such that $Sz_{n_\nu} \xrightarrow{w} u$ in X^*. Hence $z_{n_\nu} - Sz_{n_\nu} \xrightarrow{w^*} z - u$ in X^{***}. Since $z_{n_\nu} - Sz_{n_\nu} \in Y^\perp$ by (1) and $Y^\perp$ is w^* closed in X^{***}, we have $z - u \in Y^\perp$. Using the fact $X^\perp \cap X^* = \{0\}$ in X^{***}, we see $u = Sz$. This concludes the proof of w^*-w continuity of S.

It is clear that w^*-w continuity of S implies w^*-w^* continuity of S. Therefore there is an operator $T : X \to A$ such that $T^* = S$ (we regard $X^\perp$ as A^*). More precisely we have

$$\langle Tx, z \rangle = \langle x, Sz \rangle \tag{2}$$

for all $x \in X$ and $z \in X^\perp$, where $\langle u, v\rangle$ for $u \in X^{**}$ and $v \in X^{***}$ denotes the duality between X^{**} and X^{***}. Next we wish to show that T is a compact operator. Since $T^* = S$ is weakly compact, by a theorem of Gantmacher [6], T is also weakly compact. Note that the range space A of T is isomorphic to l_1 and every weakly compact set in l_1 is actually strongly compact, which can be shown by Eberlein's theorem and Schur's Lemma in l_1. Therefore, we can conclude that T is a compact operator from X to A. Hence T^{**} is a compact operator from $X^{**} = X \oplus A$ to A ($\dot{A}$ instead of A^{**} because T is compact) such that

$$\langle T^{**}(x + a), z \rangle = \langle x + a, Sz \rangle \tag{3}$$

for all $x \in X$, $a \in A$ and $z \in X^\perp$.

Now we can describe Y in terms of T^{**};

$$Y = \{x + a : x \in X, a \in A \text{ and } Tx = a - T^{**}a\}. \tag{4}$$

This is true because an element $x + a$ in X^{**} ($x \in X$, $a \in A$) belongs to $Y \Leftrightarrow x + a \in (Y^\perp)_\perp \Leftrightarrow \langle x + a, z - Sz\rangle = 0$ for all $z \in X^\perp$ (see (1)) $\Leftrightarrow$ $\langle a - Tx - T^{**}a, z\rangle = 0$ for all $z \in X^\perp$ (see (3)) $\Leftrightarrow a - Tx - T^{**}a \in (X^\perp)_\perp = X \Leftrightarrow$ $a - Tx - T^{**}a = 0$ (because $a - Tx - T^{**}a \in A$ and $A \cap X = \{0\}$ in X^{**}).

Denote the identity operator on A by I_A and $T^{**}|_A$ by T_A^{**}. Since T_A^{**} is a compact operator on A, $I_A - T_A^{**}$ is a Fredholm operator. Hence the range of $I_A - T_A^{**}$, $R(I_A - T_A^{**})$, is a closed subspace of A with finite co-dimension and the kernel of $I_A - T_A^{**}$, $N(I_A - T_A^{**})$, is a finite dimensional subspace of A. Let $X_0 = T^{-1}(R(I_A - T_A^*))$, then X_0 is a closed subspace of X with finite co-dimension, so there is a finite dimensional subspace Z of X such that $X = X_0 \oplus Z$. Let $A_0 = N(I_A - T_A^{**})$. Since A_0 is finite dimensional, there is a closed subspace B of A_0 such that $A = A_0 \oplus B$.

Our claim is that Y is complemented in X^{**};

$$X^{**} = Y \oplus Z \oplus B. \tag{5}$$

It is clear that Z and B are linearly independent. We need to show that $Y \cap (Z \oplus B) = \{0\}$. Let $y = z + b$ in $Y \cap (Z \oplus B)$. By (4), $y = x + a, x \in X_0 \subset X$ and $a \in A$. Thus, we have $x - z = b - a = 0$ because $x - z \in X$ and $b - a \in A$. We have $z = 0$ since $z \in Z$ and $z = x \in X_0$, and $a = 0$ since $x = 0$ implies $a \in A_0$ and $a = b \in B$. Thus Y, Z and B are linearly independent. Choose an element $x + a$ in X^{**} $(x \in X, a \in A)$. Since $X = X_0 \oplus Z$ and $A = A_0 \oplus B$ there are $x_0 \in X_0$ and $z_1 \in Z$ such that $x = x_0 + z_1$, and $a_0 \in A_0$ and $b \in B$ such that $a = a_0 + b$. For the element x_0 we can find $b_1 \in B$ such that

$$Tx_0 = b_1 - T^{**}b_1.$$

Therefore $x + a = x_0 + z_1 + a_0 + b = (x_0 + b_1 + a_0) + z_1 + (b - b_1)$ where $x_0 + b_1 + a_0 \in Y$ (because of (4)), $z_1 \in Z$ and $b - b_1 \in B$. This completes the proof of (5).

On the other hand, from the decompositions $X^{**} = X \oplus A$, $X = X_0 \oplus Z$ and $A = A_0 \oplus B$ we have

$$X^{**} = X_0 \oplus A_0 \oplus Z \oplus B. \tag{6}$$

By (5) and (6) we see $Y \sim X_0 \oplus A_0$. Condition c) of the hypothesis of the Theorem implies that $X \oplus C \sim X$ if C is finite dimensional, which implies that if $D \subset X$ has finite co-dimension in X then $D \sim X$. Consequently $Y \sim X_0 \oplus A_0 \sim X \oplus A_0 \sim X$. This completes the proof of the theorem.

We observe that for all compact operators $T : X \to A$, the set described in (4) gives a predual of X^*.

Professor William B. Johnson pointed out to us that if a separable dual space Y^* contains l_1 as a complemented subspace then Y^* has non-unique preduals. The argument is as follows: $Y^* \sim Y^* \oplus l_1$ implies that $Y \oplus C(K)$ is a predual of Y^* for every compact countable set K. If all these preduals were mutually isomorphic, then Y would be a universal space for the class $\{C(K)\}$ and the argument in W. Szlenk's paper [11] would imply that Y^* is non-separable. This argument shows that the space X^{**} in our Theorem does not have a unique predual. However, in this case a simpler argument is that $X^* \oplus c_0$ is a predual of X^{**} and it is not isomorphic to X^* because c_0 cannot be embedded in a separable dual space.

A natural question is whether there must exist non-isomorphic preduals of X^* if X^{**} is non-separable and X is separable. Note that the usual preduals of l_1, l_∞, $L_\infty[0, 1]$ and the X^{**} in our Theorem satisfy these conditions.

Added in proof: Recently we have proved the Theorem without using hypothesis c).

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